

The fine-tuning cost of the likelihood in SUSY models.

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Abstract

In SUSY models, the fine tuning of the electroweak (EW) scale with respect to their parameters $\gamma_i = \{m_0, m_{1/2}, \mu_0, A_0, B_0, \dots\}$ and the maximal likelihood L to fit the experimental data are usually regarded as two different problems. We show that, if one regards the EW minimum conditions as constraints that *fix* the EW scale, this commonly held view is not correct and that the likelihood contains all the information about fine-tuning. In this case we show that the corrected likelihood is equal to the ratio L/Δ of the usual likelihood L and the traditional fine tuning measure Δ of the EW scale. A similar result is obtained for the integrated likelihood over the set $\{\gamma_i\}$, that can be written as a surface integral of the ratio L/Δ , with the surface in γ_i space determined by the EW minimum constraints. As a result, a large likelihood actually demands a large ratio L/Δ or equivalently, a small $\chi_{\text{new}}^2 = \chi_{\text{old}}^2 + 2 \ln \Delta$. This shows the fine-tuning cost to the likelihood (χ_{new}^2) of the EW scale stability enforced by SUSY, that is ignored in data fits. A good $\chi_{\text{new}}^2/\text{d.o.f.} \approx 1$ thus demands SUSY models have a fine tuning amount $\Delta \ll \exp(\text{d.o.f.}/2)$, which provides a model-independent criterion for acceptable fine-tuning. Numerical methods to fit the data can easily be adapted to account for this effect.

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1 Fine tuning versus likelihood.

There is a commonly held view that the fine tuning Δ of the EW scale [1] wrt UV parameters $\gamma_i = \{m_0, m_{1/2}, \mu_0, A_0, B_0, \dots\}$ of SUSY models and the maximal likelihood L to fit the experimental data are two separate, unrelated quantities. The purpose of this letter is to show that this is not true and that there is actually a mathematical link between the likelihood and the EW scale fine-tuning. This is important given the current LHC SUSY searches and the large value of Δ in some models [2, 3], often seen as a probe against SUSY existence [4].

We show that, if one regards the EW minimum conditions as constraints that fix the EW scale, the likelihood L or its integrated form [5] over a set of parameters γ_i include the effects of the fine-tuning of the EW scale. After eliminating the dependent parameter (γ_κ) fixed by these constraints, we show that the corrected (constrained) likelihood is equal to the ratio of the usual likelihood and the traditional fine tuning measure Δ . One can also consider the integrated likelihood over a (sub)set of γ_i , and we show that this can also be expressed as a surface integral of the ratio L/Δ where Δ is the fine tuning measure (in “quadrature”) and the surface in γ_i space is determined by the two EW minimum constraints. Maximizing the likelihood then requires a large ratio L/Δ , usually favouring minimal fine-tuning Δ . Numerical methods to calculate the likelihood in the presence of EW minimum constraints can easily be adapted to account for such effects.

We start with some comments about fine tuning, Δ . Common definitions of Δ are

$$\Delta_m = \max |\Delta_{\gamma_i}|, \quad \Delta_q = \left(\sum_i \Delta_{\gamma_i}^2 \right)^{1/2}, \quad \Delta_{\gamma_i} = \frac{\partial \ln v^2}{\partial \ln \gamma_i^2}, \quad \gamma_i = m_0, m_{1/2}, \mu_0, A_0, B_0, \dots \quad (1)$$

Δ_m was the first measure used [6], but Δ_q is also used, giving very similar results [2]. These measures were based on physical intuition rather than rigorous mathematical grounds. Another drawback is that Δ provides a local measure (in the space $\{\gamma_i\}$) of the quantum cancellations that fix the EW scale, while to compare models a more global measure is desirable. It is often assumed that a solution to the hierarchy problem requires small fine tuning but there is no widely accepted upper value for Δ . A related issue is that one can have a model at a point in the parameter space $\{\gamma_i\}$ with very good³ $\chi^2/\text{d.o.f.}$ but very large Δ and another point in parameter space with good $\chi^2/\text{d.o.f.}$ but much smaller Δ . In this situation, how do we decide which of these two cases is better? The same question applies when comparing similar points from different models.

To address this question we note that to maximize the likelihood one adjusts (“tunes”) the parameters (including γ_i) to best fit the EW data. At the same time, one also must adjust the same parameters to satisfy the two EW minimum constraints, one of which actually *fixes* the EW scale vev (which is central to the definition of fine tuning). At the technical level there is no significant distinction between these constraints and the corresponding adjustment (“tuning”) of the parameters, which implies that fine tuning and likelihood must be related. A proper calculation of the latter should then account for fine tuning effects associated with EW scale stability under variations of $\{\gamma_i\}$, too. Here we quantify this effect.

³d.o.f.=number of degrees of freedom, equal to number of observables fitted minus that of parameters.

2 The relation of constrained likelihood to EW fine tuning.

Consider the scalar potential which in SUSY models has the generic form

$$\begin{aligned}
V = & m_1^2 |H_1|^2 + m_2^2 |H_2|^2 - (m_3^2 H_1 \cdot H_2 + h.c.) \\
& + (\lambda_1/2) |H_1|^4 + (\lambda_2/2) |H_2|^4 + \lambda_3 |H_1|^2 |H_2|^2 + \lambda_4 |H_1 \cdot H_2|^2 \\
& + \left[(\lambda_5/2) (H_1 \cdot H_2)^2 + \lambda_6 |H_1|^2 (H_1 \cdot H_2) + \lambda_7 |H_2|^2 (H_1 \cdot H_2) + h.c. \right]. \quad (2)
\end{aligned}$$

The couplings λ_j and soft masses m_i include radiative corrections. Introduce the notation

$$\begin{aligned}
m^2 & \equiv m_1^2 \cos^2 \beta + m_2^2 \sin^2 \beta - m_3^2 \sin 2\beta \\
\lambda & \equiv \frac{\lambda_1}{2} \cos^4 \beta + \frac{\lambda_2}{2} \sin^4 \beta + \frac{\lambda_{345}}{4} \sin^2 2\beta + \sin 2\beta (\lambda_6 \cos^2 \beta + \lambda_7 \sin^2 \beta) \quad (3)
\end{aligned}$$

and $\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5$. Using eq.(3), the minimum conditions of V take the compact form

$$v^2 + \frac{m^2}{\lambda} = 0, \quad 2\lambda \frac{\partial m^2}{\partial \beta} - m^2 \frac{\partial \lambda}{\partial \beta} = 0, \quad (4)$$

$v^2 = v_1^2 + v_2^2$ is a combination of vev's of h_1^0, h_2^0 , $\tan \beta = v_2/v_1$. The solutions to constraints (4) fix the EW scale v and $\tan \beta$ for given SUSY UV parameters $\{\gamma_i\}$. It is convenient to define

$$\begin{aligned}
f_1(\gamma_i; v, \beta, y_t, y_b, \dots) & \equiv v - \left(-\frac{m^2}{\lambda} \right)^{1/2}, \\
f_2(\gamma_i; v, \beta, y_t, y_b, \dots) & \equiv \tan \beta - \tan \beta_0(\gamma_i, v, y_t, y_b), \quad \gamma_i = \{m_0, m_{1/2}, \mu_0, A_0, B_0\}. \quad (5)
\end{aligned}$$

where β_0 denotes the root of the second eq in (4). We can therefore use the equations $f_1 = f_2 = 0$ to impose the EW constraints of eqs.(4) that relate (fix) the EW scale and $\tan \beta$ in terms of the other parameters. The arguments of $f_{1,2}$ include top, bottom Yukawa couplings, the EW scale v and $\tan \beta$ while the dots denote other parameters present at one-loop and beyond (gauge couplings, etc), that we ignore without loss of generality; γ_i shown are for the constrained MSSM, but the extension to other SUSY models is trivial. These constraints can be assumed to be factorized out of the general likelihood function of a model $L(\text{data}|\gamma_i)$, quantifying the likelihood to fit the data with given γ_i

$$L \rightarrow L \delta(f_1(\gamma_i; v, \beta, y_t, y_b)) \delta(f_2(\gamma_i; v, \beta, y_t, y_b)). \quad (6)$$

From experiment one also has accurate measurements such the masses of the top (m_t), bottom (m_b) or Z boson (m_Z). To illustrate the point and to a good approximation we implement these constraints by Dirac delta functions of suitable arguments⁴, which are again assumed to be factorized out of the general likelihood function L :

$$L \rightarrow L \delta(m_t - m_t^0) \delta(m_b - m_b^0) \delta(m_Z - m_Z^0), \quad (7)$$

where m_t^0, m_b^0, m_Z^0 are numerical values from experiment. The same can be done for other observables, such as the well measured α_{em} and α_3 gauge couplings.

⁴One can implement these via Gaussian distributions, with $\delta_\sigma(x) \rightarrow 1/(\sigma\sqrt{2\pi}) \exp(-x^2/2\sigma^2)$, $\sigma \rightarrow 0$.

2.1 The local case.

When testing a SUSY model with a given set of parameters ($\{\gamma_i\}$), one option is to marginalize (i.e. integrate) the likelihood L over unrelated, “nuisance” parameters that are determined accurately from the data. An example of such parameters are the Standard Model Yukawa couplings y_t, y_b, \dots [7], see also [5]⁵. Another option to eliminate these nuisance parameters (y_t, y_b, \dots) is to construct the profile likelihood, in which they are removed by the condition to maximise L wrt them, for fixed $\{\gamma_i\}$. In the following we integrate L over $y_{t,b}$, however, to ensure our study can be used to construct the profile likelihood, later on we also present the result without integrating over $y_{t,b}$. Further, we integrate L over the vev v (or well-measured m_Z^0) and $\tan \beta$, which are also fixed by minimization constraints (4), (6) and (7). One has

$$L(\text{data}|\gamma_i) = N_1 \int d(\ln y_t) d(\ln y_b) dv d(\tan \beta) \delta(m_Z - m_Z^0) \delta(m_t - m_t^0) \delta(m_b - m_b^0) \\ \times \delta(f_1(\gamma_i; v, \beta, y_t, y_b)) \delta(f_2(\gamma_i; v, \beta, y_t, y_b)) L(\text{data}|\gamma_i; v, \beta, y_t, y_b), \quad (8)$$

where $L(\text{data}|\gamma_i; \beta, v, y_t, y_b)$ is the likelihood to fit the data with a particular set of values for $\gamma_i, y_{t,b}$, etc, while $L(\text{data}|\gamma_i)$ is the (“constrained”) likelihood in the presence of the EW constraints and is a function of γ_i only; the associated χ^2 is given by $\chi^2 = -2 \ln L$. In eq.(8) all parameters $\gamma_i, v, \tan \beta, y_t, y_b, \dots$ are independent since the constraints that render them dependent variables are enforced by the delta functions associated to the theoretical and experimental constraints. We integrated over $\ln y_{t,b}$ instead of $y_{t,b}$ for later convenience, however the conclusion is independent of this. N_1 is a normalization constant⁶, $N_1 = m_b^0 m_t^0 m_Z^0$.

To evaluate $L(\text{data}|\gamma_i)$ we use the fact that $m_Z = g v / 2$, $m_t = y_t v \sin \beta / \sqrt{2}$ and $m_b = y_b v \cos \beta / \sqrt{2}$ and, after performing the integrals over y_t, y_b, v , one finds from (8) that

$$L(\text{data}|\gamma_i) = v_0 \int d(\tan \beta) L(\text{data}|\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)), \\ \times \delta[f_1(\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta))] \delta[f_2(\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta))] \quad (9)$$

where $g^2 \equiv g_1^2 + g_2^2$ with g_1 (g_2) the gauge coupling of U(1) (SU(2)) and

$$v_0 \equiv 2m_Z^0/g = 246 \text{ GeV}, \quad \tilde{y}_t(\beta) \equiv \sqrt{2} m_t^0 / (v_0 \sin \beta), \quad \tilde{y}_b(\beta) \equiv \sqrt{2} m_b^0 / (v_0 \cos \beta). \quad (10)$$

Note that Yukawa couplings are now functions of β only. Integrating (9) over⁷ β gives:

⁵ We integrate over $y_{t,b,\dots}$ instead of the corresponding masses since they are more fundamental, while masses are derived variables; also there is no one-to-one matching of Yukawa to masses due to $\tan \beta$ dependence.

⁶ N_1 compensates the dimensionful arguments of the three Dirac delta functions in the first line of (8). The integration over v and $\tan \beta$ must be consistent with the definition of f_1 and f_2 in (5) in that $\int dv \delta(f_1(v)) = 1$ and $\int d(\tan \beta) \delta(f_2(\tan \beta)) = 1$, so these do not generate extra normalisation factors. Finally, one could in principle choose to integrate over variables other than $v, \tan \beta$ (and fixed by the two min conditions), but then the functions $f_{1,2}$ should have appropriate form not to alter their normalisation to unity.

⁷ We use $\delta(g(x)) = \delta(x - x_0) / |g'|_{x=x_0}$ with g' the derivative wrt x evaluated in x_0 ; x_0 is the unique root of $g(x_0) = 0$; we apply this to a function $g(\beta) = f_2(\gamma_i; \beta, v_0, \tilde{y}_t(\beta), \tilde{y}_b(\beta))$ for $x \equiv \tan \beta$ with the root $\beta_0 = \beta_0(\gamma_i)$.

$$L(\text{data}|\gamma_i) = v_0 \left[L(\text{data}|\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)) \delta[f_1(\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta))] \right]_{\beta=\beta_0(\gamma_i)} \quad (11)$$

Here $\beta = \beta_0(\gamma_i)$ is the unique root of $f_2(\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)) = 0$, via which it becomes a function of γ_i . It is important to note that the argument v of f_1 was replaced by v_0 so the only arguments (variables) of f_1 are γ_i . Further, due to the presence of the delta function, one of the γ_i , hereafter denoted γ_κ , can be expressed as a function of the remaining ones, $\gamma_\kappa = \gamma_\kappa^0(\gamma_j)$, $j \neq \kappa$, where f_1 vanishes if evaluated on the set $\{\gamma_j, \gamma_\kappa^0(\gamma_j)\}$, ($j \neq \kappa$) and $L(\text{data}|\gamma_i) = L(\text{data}|\gamma_i, i \neq \kappa)$. In numerical studies one usually chooses $\gamma_\kappa = \mu_0$.

The role of the minimum conditions in fixing the EW scale is manifest in (11) and in a sense one could have started directly with this eq, but we wanted to show the derivation of its parametric dependence. The effect of $\delta(m_Z - m_Z^0)$ in (8) was to fix the variable v in (11) to the EW scale $v_0 = 246$ GeV, while the effect of $\delta(m_t - m_t^0)$ and $\delta(m_b - m_b^0)$, upon integration over $y_{t,b}$, was to bring in (11) a dependence of $\tilde{y}_{t,b}$ on β .

The result of eq.(11) has a similar structure in the absence of integrating over Yukawa couplings in (8). This is important if one wants to construct the profile likelihood, in which case Yukawa couplings are not integrated out. They remain instead independent parameters, on equal footing with $\{\gamma_i\}$, and can be adjusted to maximise the (unintegrated, constrained) likelihood, for fixed $\{\gamma_i\}$. In this case, eq.(8) is modified accordingly and gives

$$\begin{aligned} L(\text{data}|\gamma_i, y_t, y_b) &= m_Z^0 \int dv d(\tan \beta) \delta(m_Z - m_Z^0) \delta(f_1(\gamma_i; v, \beta, y_t, y_b)) \\ &\quad \times \delta(f_2(\gamma_i; v, \beta, y_t, y_b)) L(\text{data}|\gamma_i; v, \beta, y_t, y_b) \\ &= v_0 \left[L(\text{data}|\gamma_i; v_0, \beta, y_t, y_b) \delta[f_1(\gamma_i; v_0, \beta, y_t, y_b)] \right]_{\beta=\beta_0(\gamma_i; y_t, y_b)}, \end{aligned} \quad (12)$$

and the last line above is the counterpart of eq.(11). One can then construct the profile likelihood from the lhs in (12) by eliminating $y_{t,b}$ via the condition to maximise it wrt $y_{t,b}$ (for fixed $\{\gamma_i\}$). The profile likelihood is then $L(\text{data}|\gamma_i, y_t(\gamma_i), y_b(\gamma_i))$. The only difference between (11), (12) is that the latter has an extended set of independent parameters to include $y_{t,b}$. Given this similarity, the results we derive in the following are immediately extended to apply to the lhs of (12) by a simple replacement $\{\gamma_i\} \rightarrow \{\tilde{\gamma}_i\} \equiv \{\gamma_i, y_t, y_b\}$ in the steps below.

Returning to eq.(11), we use that $\gamma_\kappa = \gamma_\kappa^0(\gamma_{j \neq \kappa})$ is a solution to the constraint $f_1 = 0$, with independent γ_j ($j \neq \kappa$) fixed to some numerical values. Then eq.(11) can be written as (see footnote 7):

$$L(\text{data}|\gamma_i) = \frac{1}{\Delta_{\gamma_\kappa}} \delta(\ln \gamma_\kappa - \ln \gamma_\kappa^0(\gamma_j)) L(\text{data}|\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)) \Big|_{\beta=\beta_0(\gamma_i)}, \quad j \neq \kappa, \quad (13)$$

with⁸:

$$\Delta_{\gamma_\kappa} \equiv \left| \frac{\partial \ln \tilde{v}^2(\gamma_i; \beta_0(\gamma_i))}{\partial \ln \gamma_\kappa^2} \right|_{\gamma_\kappa = \gamma_\kappa^0(\gamma_j); j \neq \kappa}; \quad \text{where} \quad \tilde{v}(\gamma; \beta) \equiv \left(\frac{-m^2}{\lambda} \right)^{1/2} \quad (14)$$

⁸ With the arguments of f_1 as in (11), $\tilde{v}(\gamma_i; \beta) = v_0 - f_1 = (-m^2/\lambda)^{1/2}$, $\partial f_1 / \partial \gamma_\kappa = -\partial \tilde{v} / \partial \gamma_\kappa$.

where we used the first eq in (5) and the updated arguments of f_1 as shown in (11). (For the CMSSM one has $\gamma_j = \{m_0, m_{1/2}, B_0, A_0\}$, with β, μ_0 as output, the latter in the role of γ_κ).

Eq.(13) is an interesting result that shows that there exists a close relation of the likelihood to the fine tuning Δ_{γ_κ} of the EW scale wrt a parameter γ_κ (c.f. eq.(1)). The left hand side (lhs) in (13) gives the likelihood as a function of the *independent* variables $\gamma_{j \neq \kappa}$, and is suppressed by the partial fine tuning wrt γ_κ that emerges in denominator on the rhs. Before discussing this further, note that the above effect can also be seen if we formally integrate over γ_κ . Then

$$L(\text{data}|\gamma_{i \neq \kappa}) = \int d(\ln \gamma_\kappa) L(\text{data}|\gamma_i), \quad (15)$$

which gives

$$L(\text{data}|\gamma_{i \neq \kappa}) = \frac{1}{\Delta_{\gamma_\kappa}} L(\text{data}|\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)) \Big|_{\beta=\beta_0(\gamma_i); \gamma_\kappa=\gamma_\kappa^0(\gamma_j); j \neq \kappa}, \quad (16)$$

Eq.(16) shows the relation between the constrained likelihood (on the lhs) and the unconstrained likelihood (on the rhs) without the EW stability constraint. The lhs is what should be maximized when performing data fits numerically. To this purpose one must maximize the ratio of the unconstrained likelihood and the fine tuning Δ_{γ_κ} wrt γ_κ that was eliminated by an EW min condition. If $\Delta_{\gamma_\kappa} \gg 1$, it reduces considerably the corrected likelihood.

The discussion after eq.(13) has assumed that all γ_j ($j \neq \kappa$) were *fixed* and that one can solve the constraint $f_1 = 0$ in favour of $\gamma_\kappa = \gamma_\kappa^0(\gamma_j)$. One can avoid these restrictions, and obtain the constrained likelihood in a form manifestly symmetric in all γ_i . Let us denote by $\{\gamma_i^0\}$ (all i) a root of the equation that defines the surface of EW minimum $f_1 = 0$ where f_1 has the arguments displayed in (11). We denote $z_i \equiv \ln \gamma_i$, $z_i^0 \equiv \ln \gamma_i^0$ and using a short notation that only shows the dependence of f_1 on $z_i = \ln \gamma_i$ and after a Taylor expansion of $f_1(z_i)$, we have: $\delta(f_1(z_i)) = (1/|\nabla f_1|)_0 \delta[\vec{n} \cdot (\vec{z} - \vec{z}^0)]$, where $\vec{n} = (\nabla f_1/|\nabla f_1|)_0$ is the normal to this surface, ∇ is evaluated in basis \vec{z} , which has components z_1, \dots, z_n , and the subscript in $(\nabla f_1)_0$ stands for evaluation at z_i^0 for which $f_1(z_i^0) = 0$. With this, eq.(11) can be written as

$$L(\text{data}|\gamma_i) = \frac{1}{\Delta_q} \delta\left(\sum_{j \geq 1} n_j (\ln \gamma_j - \ln \gamma_j^0)\right) L(\text{data}|\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)) \Big|_{\beta=\beta_0(\gamma_i)} \quad (17)$$

where n_j are components of \vec{n} and we used that $(1/v_0)|\nabla f_1|_0 = \Delta_q$; Δ_q is the fine tuning in quadrature, defined in (1) with the replacement $v \rightarrow \tilde{v}(\gamma_i, \beta_0(\gamma_i))$ and evaluated at $\gamma_i = \gamma_i^0$ (all i). With n_j independent, eq.(17) is satisfied if all $\gamma_i = \gamma_i^0$, i.e. when $f_1 = 0$. Integrating over $\ln \gamma_\kappa$ as done in (15), recovers eq.(16). Alternatively, one can define a new “direction” (variable) $\tilde{\gamma}$ with $\ln \tilde{\gamma} = \sum_{i \geq 1} n_i \ln \gamma_i$, and integrate (17) over $d(\ln \tilde{\gamma})$ instead of $d(\ln \gamma_\kappa)$. The result of this integral is

$$L(\text{data}|\gamma_i^0) = \frac{1}{\Delta_q} L(\text{data}|\gamma_i; v_0, \beta, \tilde{y}_t(\beta), \tilde{y}_b(\beta)) \Big|_{\beta=\beta_0(\gamma_i); \gamma_i=\gamma_i^0} \quad (18)$$

This is the counterpart of (16), in a format symmetric over γ_i that allows all of them to vary simultaneously on the surface $f_1 = 0$. If all γ_i other than γ_κ are fixed, then (16) is recovered. To maximize the constrained likelihood, one should actually maximise the ratio of the unconstrained likelihood and the fine tuning Δ_q , evaluated on the surface $f_1 = 0$.

Eqs.(16), (18) have an important consequence which is a change of the value of $\chi^2/\text{d.o.f.}$ in precision data fits of the models. Let us introduce $\chi^2 = -2 \ln L$, with similar arguments, then from eq.(16), the corresponding χ^2 values of the constrained (χ_{new}^2) and unconstrained (χ_{old}^2) likelihoods are related by

$$\chi_{\text{new}}^2(\gamma_j) = \chi_{\text{old}}^2(\gamma_j; \gamma_\kappa^0(\gamma_j)) + 2 \ln \Delta_{\gamma_\kappa}(\gamma_j), \quad (j \neq \kappa) \quad (19)$$

where we made explicit the arguments of these functions, with $\gamma_\kappa = \gamma_\kappa^0(\gamma_j)$ ($j \neq \kappa$). Using instead the manifestly symmetric form of eq.(18), the corresponding χ^2 are related by

$$\chi_{\text{new}}^2(\gamma_i) = \left[\chi_{\text{old}}^2(\gamma_i) + 2 \ln \Delta_q(\gamma_i) \right]_{f_1=0} \quad (20)$$

where the subscript $f_1=0$ stresses that there is a correlation among the parameters $\{\gamma_i\}$.

The result in (20) and the previous equations remain valid if one does not integrate over Yukawa couplings as done in (8) but uses instead as a starting point the result of (12). All steps after (12) are similar, with the only difference that one must extend the set of arguments $\{\gamma_i\}$ of all functions in (20) and previous equations, to the set $\{\gamma_i, y_t, y_b\}$. In particular $\Delta_q(\gamma_i)$ of eq.(20) is replaced by $\Delta_q(\gamma_i, y_t, y_b)$ which thus includes the fine tuning wrt γ_i and y_t, y_b as well, all added “in quadrature” (i.e. the sum in the definition of Δ_q of eq.(1) is extended to include y_t, y_b). This observation is relevant when constructing the (constrained) profile likelihood function and its associated χ^2 . For this, Yukawa couplings $y_{t,b}$ are eliminated from the constrained likelihood by the condition of maximising it wrt each of them, for fixed $\{\gamma_i\}$.

Eqs.(19), (20) show that χ_{old}^2 receives a positive correction due to the fine tuning amount of the EW scale. For a realistic model one must then minimize χ_{new}^2 wrt parameters γ_j ; a good fit requires a $\chi_{\text{new}}^2/\text{d.o.f.}$ close to one. Eq.(16) to (20) are the main results of this paper.

The fine tuning correction can be significant and comparable to χ_{old}^2 . For example, for a modest fine tuning $\Delta = 10$, then $2 \ln \Delta \sim 4.6$ which is significant, while for $\Delta \sim 500$, $2 \ln \Delta \sim 12.5$. This is the “hidden” fine-tuning cost of the likelihood (χ^2) that is ignored in current calculations of this quantity in SUSY models, and is associated with the EW scale stability that supersymmetry was supposed to enforce in the first place.

Another consequence of the last two equations is that we can infer from them a model-independent value for what is considered acceptable fine tuning for any viable model. This is relevant since such value was traditionally obtained based on intuitive rather than mathematical grounds. With $\chi^2/\text{d.o.f.}$ required to be near unity, one obtains the upper bound on the fine tuning, giving $\Delta \ll \exp(\text{d.o.f.}/2)$. This concludes our discussion for the “local” case, without marginalizing over the remaining, independent parameters.

2.2 The global case.

In the following we explore a more general case by computing the integrated (i.e. “global”) likelihood over all γ_i parameters. To compare different SUSY models, that span a similar SUSY space, the γ_i -integrated likelihood can provide useful information [5]. This is something very familiar in the Bayesian approach, when computing the Bayesian evidence [7], necessary to compare the relative probability of different models. The present case of integrated likelihood is a special case of this, with a particular choice for priors. Integrating over all UV parameters γ_i , the result for the global likelihood to fit the data is

$$L(\text{data}) = \int d(\ln \gamma_1) \cdots d(\ln \gamma_n) L(\text{data}|\gamma_i), \quad (21)$$

with $\gamma_i = \{m_0, m_{1/2}, \mu_0, A_0, B_0\}$ for CMSSM. For later convenience, we integrate over $\ln \gamma_i$ rather than γ_i ; this is one possible choice of many, and relates to the question whether $\ln \gamma_i$ are more fundamental variables than γ_i or more generally, what the integral measure in $\{\gamma_i\}$ space is. The rhs of (21) can be converted into a surface integral by using the formula [8]⁹

$$\int_{R^n} h(z_1, \dots, z_n) \delta(g(z_1, \dots, z_n)) dz_1 \dots dz_n = \int_{S_{n-1}} dS_{n-1} h(z_1, \dots, z_n) \frac{1}{|\nabla_{z_i} g|}, \quad (22)$$

The surface S_{n-1} is defined by $g(z_1, \dots, z_n) = 0$ and ∇ is the gradient in basis z_i , while dS_{n-1} is the element of area on this surface. Using (22) with the replacement $z_i \rightarrow \ln \gamma_i$, we find from (11), (21)

$$\begin{aligned} L(\text{data}) &= \int d(\ln \gamma_1) \cdots d(\ln \gamma_n) L(\text{data}|\gamma_i) \\ &= \int_{f_1=0} dS_{\ln \gamma} \frac{1}{\Delta_q(\gamma)} L(\text{data}|\gamma_i; v_0, \beta_0(\gamma_i), \tilde{y}_t(\beta_0(\gamma_i)), \tilde{y}_b(\beta_0(\gamma_i))) \end{aligned} \quad (23)$$

where $dS_{\ln \gamma}$ is the element of area in the parameter space $\{\ln \gamma_i\}$. The last integral is over a surface in $\{\gamma_i\}$ space, given by EW minimum equation $f_1 = 0$ (with $f_2 = 0$ or equivalently $\beta = \beta_0(\gamma_i)$). In the last eq we denoted

$$\Delta_q(\gamma) \equiv |\nabla_{\ln \gamma_i} \ln \tilde{v}(\gamma_i; \beta_0(\gamma_i))| = \left(\sum_{j \geq 1} \Delta_{\gamma_j}^2 \right)^{1/2}, \quad \Delta_{\gamma_j} \equiv \left| \frac{\partial \ln \tilde{v}^2(\gamma_i; \beta_0(\gamma_i))}{\partial \ln \gamma_j^2} \right|, \quad (24)$$

with $\gamma_j \equiv m_0, m_{1/2}, \mu_0, A_0, B_0$ and where $\nabla_{\ln \gamma_i}$ is the gradient in the coordinate space¹⁰ $\{\ln \gamma_i\}$ and $\Delta_q(\gamma)$ is the fine tuning in quadrature, γ_i -dependent.

Eq.(23) with (24) is a global version of eq.(18) and shows a similar, interesting result. It presents the mathematical origin and the role of the traditional fine tuning measure, which

⁹ Eq.(22) can be derived using the discussion around eq.(17) and the definition of the element area dS_{n-1} .

¹⁰ Another form of (23) is found by replacing dS, ∇ by their values in $\{\gamma_i\}$ space (instead of $\{\ln \gamma_i\}$) and dividing by the product $\gamma_1 \dots \gamma_n$ under integral (23). Δ_{γ_j} is replaced by a derivative wrt γ_j instead of $\ln \gamma_j$.

was not introduced ad-hoc but turned out to be an intrinsic part of the likelihood function in the presence of the EW minimum constraints. The result of (23) is useful when comparing the relative probability of two models.

The consequence of the EW scale stability constraint is that one should maximize in this case the integral of the ratio L/Δ . Changing the measure of integration over γ_i also changes the likelihood “flux” under the surface integral, but the overall suppression by Δ remains. In a Bayesian interpretation, changing the measure under the integral corresponds to taking different priors for the variables γ_i that were marginalized. The measure we took corresponds to log priors in the Bayesian approach. For a related Bayesian interpretation of these results and in particular the integrated likelihood see [2, 7].

2.3 Phenomenological implications.

The effects we identified have phenomenological consequences. Very often, in SUSY models a good likelihood fit to the EW data (i.e. small $\chi^2/\text{d.o.f.}$) usually prefers values for the higgs mass m_h that have smaller quantum corrections, with m_h near the LEP2 lower bound. Further, in models such as the constrained MSSM (CMSSM), the CMSSM with relaxed, non-universal higgs soft masses or the CMSSM with non-universal gaugino masses, the fine tuning grows approximately exponentially with the higgs mass, to large values, of order $\Delta \sim 500 - 1000$ [2] for a Higgs mass near the observed value of ≈ 125 GeV [9]. In such models the constrained likelihood $L(\text{data}|\gamma_i)/\Delta$ is significantly smaller. This is seen from eqs.(19), (20) showing an increase of $\chi^2/\text{d.o.f.}$ by $2 \ln \Delta/\text{d.o.f.}$.

For example¹¹, in the CMSSM model with $m_h \approx 125$ GeV and taking a minimal, optimistic value $\Delta \approx 100$, then $\delta\chi^2/\text{d.o.f.} \approx 9.2/9$ which is a very large correction to χ^2 , while taking the value $\Delta \approx 500$, $\delta\chi^2/\text{d.o.f.} \approx 12.5/9$. Even for models that can have $\Delta \approx 10$, $\delta\chi^2/\text{d.o.f.} \approx 4.6/\text{d.o.f.}$ which is significant for comparable d.o.f. From this it is clear that the determination of a large fine tuning can rule out models even before a detailed, traditional χ^2 analysis is undertaken! In the light of these results, searching for SUSY models with small $\Delta < \mathcal{O}(10)$ [3] for the observed value of m_h is well-motivated.

3 Conclusions

There exists a commonly held view that the likelihood to fit the data within a SUSY model and the familiar fine-tuning measure Δ of the EW scale are two distinct problems that should be treated separately. In this paper we have shown that this traditional view is not correct and that there is a strong, mathematical link between these two problems, once the EW minimum conditions are regarded as constraints of the model that fix the EW scale vev. Imposing these constraints, the emerging constrained likelihood $L(\text{data}|\gamma_i)$ is proportional to the ratio L/Δ of the usual, unconstrained likelihood L and the traditional fine tuning measure Δ . A similar result applies for the integrated likelihood over the parameter space.

¹¹For a recent, detailed discussion of the $\chi^2/\text{d.o.f.}$ values in CMSSM and other models see [10]. In the examples we give we take $\text{d.o.f.}=9$ for the CMSSM, corresponding to 13 observables and 4 parameters.

Maximising the constrained likelihood determines how to compare different points in the parameter space of a model: one which has a very good L but large Δ , and another with good L but much smaller Δ .

An important consequence of the above result is that the corresponding values of χ^2 of the constrained (χ_{new}^2) and unconstrained (χ_{old}^2) likelihoods are related by the formula $\chi_{\text{new}}^2 = \chi_{\text{old}}^2 + 2 \ln \Delta$. The minimum $\chi_{\text{new}}^2/\text{d.o.f.}$, contains a positive correction which has a negative impact on the overall data fit of the model. This is the “hidden” fine-tuning cost to the likelihood/ χ^2 that is ignored in current fits of SUSY models, and is associated with the EW scale stability that low-energy supersymmetry was introduced to enforce. An acceptable upper bound of the fine tuning is given by $\Delta \ll \exp(\text{d.o.f.}/2)$, such that $\chi_{\text{new}}^2/\text{d.o.f.}$ is not significantly worse. Generically this requires $\Delta < \mathcal{O}(10)$. If this bound is not respected, one can even rule out a model without a detailed χ^2 analysis.

In conclusion, we have shown that the EW scale fine tuning does play a role in establishing if a model is realistic or not in a probabilistic sense. Current numerical methods to fit the data can easily be adapted to account for this effect.

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